

# Admissible decomposition for spectral multipliers on Gaussian $L^p$

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**ABSTRACT.** This paper concerns harmonic analysis of the Ornstein–Uhlenbeck operator  $L$  on the Euclidean space. We examine the method of decomposing a spectral multiplier  $\phi(L)$  into three parts according to the notion of admissibility, which quantifies the doubling behaviour of the underlying Gaussian measure  $\gamma$ . We prove that the above-mentioned admissible decomposition is bounded in  $L^p(\gamma)$  for  $1 < p \leq 2$  in a certain sense involving the Gaussian conical square function. The proof relates admissibility with E. Nelson’s hypercontractivity theorem in a novel way.

## 1. Introduction

**1.1. General background.** This article is a continuation of [8], regarding analysis of the *Ornstein–Uhlenbeck operator*

$$L = -\frac{1}{2}\Delta + x \cdot \nabla,$$

which on the Euclidean space  $\mathbb{R}^n$  is associated with the *Gaussian measure*

$$d\gamma(x) = \pi^{-n/2} e^{-|x|^2} dx.$$

In [8], a certain class of spectral multipliers  $\phi(L)$  was studied by means of an *admissible decomposition* — an integral representation, which takes into account the non-doubling behaviour of  $\gamma$ . This representation allows us to express the multiplier as a sum of three parts (admissible, intermediate, and non-admissible):

$$\phi(L)f = c(\pi_1 u + \pi_2 f + \pi_3 f),$$

where  $c$  is a constant and  $u$  arises from  $f$ . An  $L^1$ -estimate was then obtained in terms of an *admissible conical square function*  $Sf$ , namely,

$$\|\pi_1 u\|_1 \lesssim \|Sf\|_1, \quad \|\pi_2 f\|_1 \lesssim \|f\|_1, \quad \|\pi_3 f\|_1 \lesssim \|(1 + \log_+ |\cdot|) Mf\|_1,$$

but the third estimate with a logarithmic weight and a maximal function  $Mf$  is clearly unsatisfactory. This shortcoming calls into question whether the admissible decomposition is at all suitable for studying boundedness of spectral multipliers. On the other hand, such problems do not seem to appear in [14], from which the decomposition originates in connection with the Riesz transform  $\nabla L^{-1/2}$ .

The role of this article is to justify the above-mentioned approach, and to serve as an intermediate step towards a fully satisfactory  $L^1$ -estimate. Indeed, we show here that for  $1 < p \leq 2$  we have

$$\|\pi_1 u\|_p \lesssim \|Sf\|_p, \quad \|\pi_2 f\|_p \lesssim \|f\|_p, \quad \|\pi_3 f\|_p \lesssim \|f\|_p.$$

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Interestingly, the proof of the third estimate invokes the hypercontractivity theorem of E. Nelson [13], and relies on its subtle interplay with the concept of admissibility. The ultimate aim of this square function approach is to provide a metric theory of Gaussian Hardy spaces to complement the existing atomic theory [11].

**1.2. Admissible conical square function.** Recall that the *admissibility function*

$$m(x) = \min(1, |x|^{-1}), \quad x \in \mathbb{R}^n,$$

quantifies the extent to which  $\gamma$  is doubling:

$$\gamma(B(x, 2t)) \lesssim \gamma(B(x, t)), \quad t \leq m(x).$$

See [11, 10, 1] for more details.

The admissible conical square function is then defined by

$$Sf(x) = \left( \int_0^{2m(x)} \frac{1}{\gamma(B(x, t))} \int_{B(x, t)} |t^2 L e^{-t^2 L} f(y)|^2 d\gamma(y) \frac{dt}{t} \right)^{1/2}, \quad x \in \mathbb{R}^n,$$

where the diffusion semigroup

$$e^{-tL} f(x) = \int_{\mathbb{R}^n} M_t(x, y) f(y) d\gamma(y), \quad t > 0,$$

is given by the *Mehler kernel*

$$M_t(x, y) = \frac{1}{(1 - e^{-2t})^{n/2}} \exp \left( - \frac{e^{-t}}{1 - e^{-2t}} |x - y|^2 \right) \exp \left( \frac{e^{-t}}{1 + e^{-t}} (|x|^2 + |y|^2) \right).$$

The origins of this Gaussian square function can be found in [9, 14]. The Ornstein–Uhlenbeck semigroup  $(e^{-tL})_{t>0}$  is a prototypical example of a symmetric, contractive, and conservative diffusion semigroup in the sense of [16]. For more information, see the (old, but not obsolete) survey [15].

**1.3. Class of spectral multipliers.** We will consider spectral multipliers of the form

$$\phi(\lambda) = \int_0^\infty \Phi(t) (t\lambda)^2 e^{-t\lambda} \frac{dt}{t}, \quad \lambda \geq 0,$$

where  $\Phi : (0, \infty) \rightarrow \mathbb{C}$  is twice continuously differentiable and satisfies

$$(1) \quad \sup_{0 < t < \infty} (|\Phi(t)| + t|\Phi'(t)|) + \sup_{0 < t \leq 1} |t^2 \Phi''(t)| < \infty.$$

As explained in [8], these are a special kind of ‘Laplace transform type’ multipliers.

Moreover, we will refer to the following two extra conditions.

- Condition D:

$$\int_1^\infty (|\Phi'(t)| + t|\Phi''(t)|) dt < \infty.$$

- Condition P: There exists an integer  $N$  such that

$$|\Phi'(t)| + t|\Phi''(t)| \lesssim t^N, \quad t \geq 1.$$

Notice, however, that the main result is already interesting for the prototypical *imaginary powers*  $\phi(L) = L^{i\tau}$ ,  $\tau \in \mathbb{R}$ , with  $\Phi(t) = t^{-i\tau}/\Gamma(2 - i\tau)$  (or for their damped versions with  $\Phi(t) = t^{-i\tau}\chi(t)$ , where  $\chi$  is a smooth cutoff with  $1_{(0,1]} \leq \chi \leq 1_{(0,2]}$ ).

**1.4. Admissible decomposition.** The analysis is greatly simplified by switching to the discretized version of the admissibility function

$$\tilde{m}(x) = \begin{cases} 1, & |x| < 1, \\ 2^{-k}, & 2^{k-1} \leq |x| < 2^k, \quad k \geq 1, \end{cases}$$

and to the associated *admissible region*  $D = \{(y, t) \in \mathbb{R}^n \times (0, \infty) : 0 < t < \tilde{m}(y)\}$ .

Let then  $\phi$  and  $\Phi$  be as in Subsection 1.3 and let  $f$  be a polynomial with  $\int f d\gamma = 0$ . The special form of our spectral multipliers allows us to use the following integral representation with  $\delta, \delta' > 0$  and  $\kappa \geq 1$ :

$$\begin{aligned} \phi(L)f &= c_{\delta, \delta'} \int_0^\infty \Phi((\delta' + \delta)t^2)(t^2 L)^2 e^{-(\delta' + \delta)t^2 L} f \frac{dt}{t} \\ &= c_{\delta, \delta'} \left( \int_0^{\tilde{m}(\cdot)/\kappa} \tilde{\Phi}(t^2) t^2 L e^{-\delta' t^2 L} u(\cdot, t) \frac{dt}{t} \right. \\ (2) \quad &\quad + \int_0^{\tilde{m}(\cdot)/\kappa} \tilde{\Phi}(t^2) t^2 L e^{-\delta' t^2 L} (1_{D^c}(\cdot, t) t^2 L e^{-\delta t^2 L} f) \frac{dt}{t} \\ &\quad \left. + \int_{\tilde{m}(\cdot)/\kappa}^\infty \tilde{\Phi}(t^2) (t^2 L)^2 e^{-(\delta' + \delta)t^2 L} f \frac{dt}{t} \right) \\ &=: c_{\delta, \delta'} (\pi_1 u + \pi_2 f + \pi_3 f), \end{aligned}$$

where  $u(\cdot, t) = 1_D(\cdot, t) t^2 L e^{-\delta t^2 L} f$  and  $\tilde{\Phi}(t) = \Phi((\delta' + \delta)t)$ . The role of the technical parameters  $\delta, \delta'$  and  $\kappa$  is more visible in [8] than in this paper.

**1.5. Main result.** The first part of Proposition 5 refines the previous analysis of  $\pi_3$  from [8], and shows that the maximal operator

$$Mf(x) = \sup_{\varepsilon m(x)^2 < t \leq 1} |e^{-tL} f(x)|, \quad x \in \mathbb{R}^n,$$

can be disposed of, i.e. that

$$\|\pi_3 f\|_1 \lesssim \|(1 + \log_+ |\cdot|) f\|_1,$$

for multipliers satisfying Condition D. As a consequence, for all  $f \in L^1(\gamma)$  it then holds that

$$\|\phi(L)f\|_1 \lesssim \|Sf\|_1 + \|(1 + \log_+ |\cdot|) f\|_1.$$

The second part of Proposition 5 (together with Propositions 2 and 4) leads to the main result of the article:

**THEOREM.** *Let  $1 < p \leq 2$ . For multipliers satisfying Condition P, there exist values of parameters  $\delta, \delta'$  and  $\kappa$  so that*

$$\|\pi_1 u\|_p \lesssim \|Sf\|_p, \quad \|\pi_2 f\|_p \lesssim \|f\|_p, \quad \|\pi_3 f\|_p \lesssim \|f\|_p.$$

**COROLLARY.** *Let  $1 < p \leq 2$ . For spectral multipliers  $\phi$  of Subsection 1.3 satisfying Condition P we have*

$$\|\phi(L)f\|_p \lesssim \|Sf\|_p + \|f\|_p.$$

Such spectral multipliers are well known to be bounded on  $L^p(\gamma)$  for all  $1 < p < \infty$ , also in vastly more general settings [16, 4, 3]. The vertical square function that is typically used in their analysis seems, however, to be somewhat ill-suited for  $p = 1$  and the corresponding Hardy space theory. Developments of an abstract semigroup approach to Hardy spaces nevertheless exist, see [12, 7]. Recall also the relations between vertical and conical objects in [2, Proposition 2.1], showing how conical square functions dominate the vertical ones for  $p \leq 2$ . Moreover, it is curious to note that a *local* square function such as ours is sufficient for the analysis of an operator with a spectral gap (between the lowest two eigenvalues in

$\sigma(L) = \{0, 1, 2, \dots\}$ . The intriguing question whether  $\|Sf\|_p \lesssim \|f\|_p$  for  $p > 1$  is a topic of ongoing research.

## 2. Proof

Throughout the proof we assume that  $f$  is a polynomial with  $\int f d\gamma = 0$ , and therefore a finite linear combination of *Hermite polynomials* — the eigenfunctions of  $L$ . The three parts of the admissible decomposition (2) are studied separately in the following three subsections.

**2.1. Admissible part.** Let us first recall the definition of tent spaces (see [1, 10]).

DEFINITION. Let  $1 \leq p \leq 2$ . The *Gaussian tent space*  $\mathfrak{t}^p(\gamma)$  is defined to consist of functions  $u$  on the admissible region  $D = \{(y, t) \in \mathbb{R}^n \times (0, \infty) : 0 < t < \tilde{m}(y)\}$  for which

$$\|u\|_{\mathfrak{t}^p(\gamma)} = \left( \int_{\mathbb{R}^n} \left( \iint_{\Gamma(x)} |u(y, t)|^2 \frac{d\gamma(y) dt}{t\gamma(B(y, t))} \right)^{p/2} d\gamma(x) \right)^{1/p} < \infty.$$

Here  $\Gamma(x) = \{(y, t) \in D : |y - x| < t\}$  is the admissible cone at  $x \in \mathbb{R}^n$ .

Consider the admissible part

$$\pi_1 u = \int_0^{\tilde{m}(\cdot)/\kappa} \tilde{\Phi}(t^2) t^2 L e^{-\delta' t^2 L} u(\cdot, t) \frac{dt}{t}$$

for functions  $u$  in a Gaussian tent space.

Curiously, due to the non-uniformity of the admissibility function, the case  $p = 2$  is not quite as straightforward as one might expect.

PROPOSITION 1. For  $\kappa \geq 1$  and  $0 < \delta' \leq 1$  we have  $\|\pi_1 u\|_2 \lesssim \|u\|_{\mathfrak{t}^2(\gamma)}$ .

PROOF. The proof does not rely on admissibility in the sense that  $\tilde{m}(x)/\kappa$  can be replaced by any function with values in  $(0, 1]$ . Hence we may abbreviate  $\tilde{m}(x)/\kappa = m(x)$ .

Write  $\chi_t(x) = 1_{(0, m(x))}(t)$ . Given a  $g \in L^2(\gamma)$ , we argue by duality:

$$\begin{aligned} |\langle \pi_1 u, g \rangle| &= \left| \int_{\mathbb{R}^n} \int_0^{m(\cdot)} \tilde{\Phi}(t^2) t^2 L e^{-\delta' t^2 L} u(\cdot, t) \frac{dt}{t} g d\gamma \right| \\ &= \left| \int_0^1 \tilde{\Phi}(t^2) \int_{\mathbb{R}^n} t^2 L e^{-\delta' t^2 L} u(\cdot, t) \chi_t g d\gamma \frac{dt}{t} \right| \\ &= \left| \int_0^1 \int_{\mathbb{R}^n} u(\cdot, t) t^2 L e^{-\delta' t^2 L} (\chi_t g) d\gamma \frac{dt}{t} \right| \\ &\leq \|u\|_{\mathfrak{t}^2(\gamma)} \left( \int_0^1 \|t^2 L e^{-\delta' t^2 L} (\chi_t g)\|_2^2 \frac{dt}{t} \right)^{1/2}, \end{aligned}$$

and so it suffices to show that

$$\left( \int_0^1 \|t^2 L e^{-\delta' t^2 L} (\chi_t g)\|_2^2 \frac{dt}{t} \right)^{1/2} \lesssim \|g\|_2.$$

Now the uniform  $L^2$ -boundedness of  $(t^2 L)^{1/2} e^{-\frac{\delta'}{2} t^2 L}$  guarantees that

$$\begin{aligned} \|t^2 L e^{-\delta' t^2 L} (\chi_t g)\|_2 &\lesssim \|(t^2 L)^{1/2} e^{-\frac{\delta'}{2} t^2 L} (\chi_t g)\|_2 \\ &= \int_{\mathbb{R}^n} t^2 L e^{-\delta' t^2 L} (\chi_t g) \overline{\chi_t g} d\gamma, \end{aligned}$$

where the last step relies on the self-adjointness and non-negativity of  $(t^2 L)^{1/2} e^{-\delta' t^2 L}$ . Expressing  $t^2 L e^{-\delta' t^2 L}$  in terms of the kernel  $(2\delta')^{-1} t \partial_t M_{\delta' t^2}(x, y)$  we therefore see that

$$\begin{aligned}
& \int_0^1 \int_{\mathbb{R}^n} |t^2 L e^{-\delta' t^2 L}(\chi_t g)|^2 d\gamma \frac{dt}{t} \\
& \lesssim \left| \int_0^1 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} t \partial_t M_{\delta' t^2}(x, y) 1_{(0, m(y))}(t) g(y) d\gamma(y) 1_{(0, m(x))}(t) \overline{g(x)} d\gamma(x) \frac{dt}{t} \right| \\
& = \left| \int_{\mathbb{R}^n} \overline{g(x)} \int_{\mathbb{R}^n} g(y) \int_0^{m(x) \wedge m(y)} \partial_t M_{\delta' t^2}(x, y) dt d\gamma(y) d\gamma(x) \right| \\
& = \left| \int_{\mathbb{R}^n} \overline{g(x)} \int_{\mathbb{R}^n} M_{\delta' (m(x) \wedge m(y))^2}(x, y) g(y) d\gamma(y) d\gamma(x) \right| \\
& \leq \left| \int_{\mathbb{R}^n} \overline{g(x)} \int_{\{y: m(y) \leq m(x)\}} M_{\delta' m(y)^2}(x, y) g(y) d\gamma(y) d\gamma(x) \right| \\
& \quad + \left| \int_{\mathbb{R}^n} g(y) \int_{\{x: m(x) \leq m(y)\}} M_{\delta' m(x)^2}(y, x) \overline{g(x)} d\gamma(x) d\gamma(y) \right| \\
& \leq \int_{\mathbb{R}^n} |g(x)| \sup_{t>0} e^{-tL} |g|(x) d\gamma(x) + \int_{\mathbb{R}^n} |g(y)| \sup_{t>0} e^{-tL} |g|(y) d\gamma(y) \\
& \leq 2 \int_{\mathbb{R}^n} (\sup_{t>0} e^{-tL} |g|)^2 d\gamma \\
& \lesssim \|g\|_2^2,
\end{aligned}$$

where in the last step we made use of the maximal inequality. This finishes the proof.  $\square$

**PROPOSITION 2.** *Let  $1 < p \leq 2$ . For  $\kappa \geq 1$  and sufficiently small  $\delta' > 0$ , we have  $\|\pi_1 u\|_p \lesssim \|u\|_{\mathfrak{P}(\gamma)}$ . Moreover, for  $0 < \delta \leq 1$ , the function  $u(\cdot, t) = 1_D(\cdot, t) t^2 L e^{-\delta t^2 L} f$  satisfies  $\|u\|_{\mathfrak{P}(\gamma)} \lesssim \|Sf\|_p$ .*

**PROOF.** The first part of the statement follows by interpolation of Gaussian tent spaces [1, Theorem 3.3 and Corollary 3.5]. Indeed,

$$\pi_1 \text{ is bounded } \begin{cases} \mathfrak{t}^2(\gamma) \rightarrow L^2(\gamma) & \text{(by Proposition 1 above),} \\ \mathfrak{t}^1(\gamma) \rightarrow L^1(\gamma) & \text{(by [8, Proposition 2]).} \end{cases}$$

Therefore,  $\pi_1$  is also bounded  $\mathfrak{t}^p(\gamma) \rightarrow L^p(\gamma)$ , i.e.  $\|\pi_1 u\|_p \lesssim \|u\|_{\mathfrak{P}(\gamma)}$ .

The second part of the statement follows by a straightforward modification of the corresponding argument in [8, Proposition 2]. Indeed, by change of aperture on  $\mathfrak{t}^p(\gamma)$  (see [1, Theorem 3.3]) we obtain

$$\|u\|_{\mathfrak{P}(\gamma)} \lesssim \left( \int_{\mathbb{R}^n} \left( \iint_{\Gamma(x) \cap D'} |s^2 L e^{-s^2 L} f(y)|^2 \frac{d\gamma(y) ds}{s \gamma(B(y, s))} \right)^{p/2} d\gamma(x) \right)^{1/p},$$

where  $D' = \{(y, s) \in \mathbb{R}^n \times (0, \infty) : s < \sqrt{\delta} \tilde{m}(y)\}$ . The desired estimate  $\|u\|_{\mathfrak{P}(\gamma)} \lesssim \|Sf\|_p$  now follows from the pointwise inequality (see [8, Proposition 2])

$$\begin{aligned}
& \iint_{\Gamma(x) \cap D'} |s^2 L e^{-s^2 L} f(y)|^2 \frac{d\gamma(y) ds}{s \gamma(B(y, s))} \\
& \lesssim \int_0^{2m(x)} \frac{1}{\gamma(B(x, s))} \int_{B(x, s)} |s^2 L e^{-s^2 L} f(y)|^2 d\gamma(y) \frac{ds}{s}, \quad x \in \mathbb{R}^n.
\end{aligned}$$

$\square$

**2.2. Intermediate part.** Let us begin by presenting two  $L^p$ -estimates for the operators  $tLe^{-tL}$ .

LEMMA 3. *The family  $(tLe^{-tL})_{t>0}$  is uniformly bounded on  $L^p(\gamma)$  for all  $p > 1$ , that is,*

$$\sup_{t>0} \|tLe^{-tL}\|_{p \rightarrow p} < \infty.$$

Moreover, for  $1 \leq p \leq 2$  we have

$$\|1_{E'} tLe^{-tL} 1_E\|_{p \rightarrow p} \lesssim t^{-n/2} \exp\left(-\frac{d(E, E')^2}{8t}\right) \sup_{\substack{x \in E \\ y \in E'}} \exp\left(\frac{|x|^2 + |y|^2}{2}\right), \quad 0 < t \leq 1,$$

whenever  $E, E' \subset \mathbb{R}^n$ .

PROOF. The boundedness of  $tLe^{-tL}$  on  $L^p(\gamma)$  (when  $p > 1$ ) is the content of [6, Theorem 5.41], and the uniformity in  $t > 0$  follows by careful inspection of the proof.

The off-diagonal estimate for  $1_{E'} tLe^{-tL} 1_E$  is an immediate consequence of [8, Lemma 3] and follows by Hölder's inequality.  $\square$

Let us then turn to

$$\pi_2 f = \int_0^{\tilde{m}(\cdot)/\kappa} \tilde{\Phi}(t^2) t^2 Le^{-\delta' t^2 L} (1_{D^c}(\cdot, t) t^2 Le^{-\delta t^2 L} f) \frac{dt}{t}.$$

PROPOSITION 4. *Let  $1 < p \leq 2$ . For  $\kappa \geq 4$  and sufficiently small  $\delta, \delta' > 0$  we have  $\|\pi_2 f\|_p \lesssim \|f\|_p$ .*

PROOF. As in [8, Proposition 5] we have

$$(3) \quad \|\pi_2 f\|_p \lesssim \sum_{k=2}^{\infty} \sum_{l=1}^{\infty} \int_{2^{-k-1}}^{2^{-k}} \|1_{B(0, 2^{k-2})} t^2 Le^{-\delta' t^2 L} (1_{C_{k+l-1}} t^2 Le^{-\delta t^2 L} f)\|_p \frac{dt}{t},$$

where  $C_{k+l-1} := B(0, 2^{k+l-1}) \setminus B(0, 2^{k+l-2})$ .

The distance between  $B(0, 2^{k-2})$  and  $C_{k+l-1}$  is at least  $2^{k+l-3}$ . We make use of Lemma 3 to see that, for  $2^{-k-1} < t \leq 2^{-k}$  we have

$$\begin{aligned} & \|1_{B(0, 2^{k-2})} t^2 Le^{-\delta' t^2 L} (1_{C_{k+l-1}} t^2 Le^{-\delta t^2 L} f)\|_p \\ & \lesssim t^{-n} \exp\left(-\frac{4^{k+l-3}}{8\delta' t^2}\right) \exp\left(\frac{4^{k-2} + 4^{k+l-1}}{2}\right) \|t^2 Le^{-\delta t^2 L} f\|_p \\ & \lesssim 2^{kn} \exp\left(-\frac{4^{2k+l-5}}{\delta'} + 4^{k+l-1}\right) \|f\|_p \\ & \lesssim \exp(-4^{k+l}) \|f\|_p, \end{aligned}$$

when  $\delta' < 4^{-3}$ .

The right-hand side of (3) is therefore dominated by

$$\sum_{k=2}^{\infty} \sum_{l=1}^{\infty} \exp(-4^{k+l}) \|f\|_p \int_{2^{-k-1}}^{2^{-k}} \frac{dt}{t} \lesssim \|f\|_p.$$

$\square$

Notice that for  $p > 1$  the proof was simpler than for  $p = 1$  because of the uniform  $L^p$ -boundedness of  $tLe^{-tL}$ .

**2.3. Non-admissible part.** We begin by recalling the following key result. See [6, Chapter V] and [5] for more references.

**HYPERCONTRACTIVITY THEOREM** (E. Nelson [13]). *Let  $1 < p \leq q < \infty$ . Then*

$$\|e^{-tL}\|_{p \rightarrow q} \leq 1, \quad \text{whenever } t \geq \frac{1}{2} \log \frac{q-1}{p-1}.$$

Let us remark, that most proofs of this result use a different scaling/normalization of the Gaussian measure. The easiest way to convince oneself of the validity of this version is probably by the equivalence between hypercontractivity and a logarithmic Sobolev inequality (see [5]). Also note that our  $L$  is ‘one half’ of a usual Dirichlet form operator.

The following reformulation of the hypercontractivity theorem will be convenient for us:

*Let  $p > 1$ . Then for any  $t > 0$ ,*

$$(4) \quad \|e^{-tL}\|_{p \rightarrow q(t)} \leq 1, \quad \text{with the hypercontractive exponent } q(t) = 1 + (p-1)e^{2t}.$$

Finally, let us consider

$$\pi_3 f = \int_{\tilde{m}(\cdot)/\kappa}^{\infty} \tilde{\Phi}(t^2) (t^2 L)^2 e^{-(\delta'+\delta)t^2 L} f \frac{dt}{t}.$$

**PROPOSITION 5.** *For sufficiently small  $\delta, \delta' > 0$  and large enough  $\kappa$  we have:*

- *If  $\Phi$  satisfies Condition D, then  $\|\pi_3 f\|_1 \lesssim \|(1 + \log_+ |\cdot|) f\|_1$ .*
- *If  $\Phi$  satisfies Condition P, then  $\|\pi_3 f\|_p \lesssim \|f\|_p$  for  $1 < p \leq 2$ .*

**PROOF.** We will consider the two statements side by side.

**Part I:** Recall the pointwise estimate from [8, Proposition 7]:

$$(5) \quad \begin{aligned} |\pi_3 f| &\lesssim \sup_{t>0} |\Phi(t)| \left| (tL e^{-(\delta'+\delta)tL} f) \Big|_{t=\tilde{m}(\cdot)^2/\kappa^2} \right| \\ &\quad + \sup_{t>0} (|\Phi(t)| + t|\Phi'(t)|) \left| (e^{-(\delta'+\delta)tL} f) \Big|_{t=\tilde{m}(\cdot)^2/\kappa^2} \right| \\ &\quad + \int_{\tilde{m}(\cdot)^2/\kappa^2}^{\infty} (|\Phi'(t)| + t|\Phi''(t)|) |e^{-(\delta'+\delta)tL} f| dt. \end{aligned}$$

We will estimate the  $L^p$ -norms of the three terms on the right-hand side separately.

For  $1 \leq p \leq 2$  and  $\kappa$  large enough we have

$$\left\| (tL e^{-(\delta'+\delta)tL} f) \Big|_{t=\tilde{m}(\cdot)^2/\kappa^2} \right\|_p + \left\| (e^{-(\delta'+\delta)tL} f) \Big|_{t=\tilde{m}(\cdot)^2/\kappa^2} \right\|_p \lesssim \|f\|_p,$$

by an immediate generalization of [8, Lemma 6]. Together with the general assumption (1) on  $\Phi$ , this takes care of the first two terms of (5).

The range  $\int_1^{\infty} dt$  in the third term in (5) is dealt with Conditions D and P separately. For  $p = 1$ , Condition D guarantees that

$$\int_1^{\infty} (|\Phi'(t)| + t|\Phi''(t)|) \|e^{-(\delta'+\delta)tL} f\|_1 dt \lesssim \|f\|_1.$$

For  $1 < p \leq 2$  we may use interpolation to see that  $\|e^{-tL} f\|_p \lesssim e^{-\theta_p t} \|f\|_p$  (recall that  $f$  is a polynomial with zero mean). Indeed, denoting by  $E_0$  the spectral projection onto the kernel of  $L$ , we have  $\|e^{-tL}(I - E_0)\|_{2 \rightarrow 2} = e^{-t}$  and  $\|e^{-tL}(I - E_0)\|_{1 \rightarrow 1} \leq 1$ . Hence we obtain the claim with  $\theta_p = 2 - 2/p$ . Now Condition P implies that

$$\int_1^{\infty} (|\Phi'(t)| + t|\Phi''(t)|) \|e^{-(\delta'+\delta)tL} f\|_p dt \lesssim \left( \int_1^{\infty} t^N e^{-\theta_p(\delta'+\delta)t} dt \right) \|f\|_p \lesssim \|f\|_p.$$

It remains to consider the range  $\int_{\tilde{m}(\cdot)^2/\kappa^2}^1 dt$  in the third term in (5). By the assumption (1),  $\sup_{0 < t \leq 1} (t|\Phi'(t)| + t^2|\Phi''(t)|) < \infty$ , and so for  $1 \leq p \leq 2$  we have

$$\left\| \int_{\tilde{m}(\cdot)^2/\kappa^2}^1 (|\Phi'(t)| + t|\Phi''(t)|) |e^{-(\delta'+\delta)tL} f| dt \right\|_p \lesssim \left\| \int_{\tilde{m}(\cdot)^2/\kappa^2}^1 |e^{-(\delta'+\delta)tL} f| \frac{dt}{t} \right\|_p,$$

which is analyzed further below.

**Part II (setup):** We will then examine the remaining integral over annuli and separate the off-diagonal and on-diagonal parts. More precisely, let us write  $C_0 = B(0, 1)$  and  $C_k = B(0, 2^k) \setminus B(0, 2^{k-1})$  for  $k \geq 1$ , and let  $C_0^* = B(0, 2)$ ,  $C_1^* = B(0, 4)$ , and  $C_k^* = B(0, 2^{k+1}) \setminus B(0, 2^{k-2})$  for  $k \geq 2$ . Then, for  $1 \leq p \leq 2$ , we have

$$\begin{aligned} \left\| \int_{\tilde{m}(\cdot)^2/\kappa^2}^1 |e^{-(\delta'+\delta)tL} f| \frac{dt}{t} \right\|_p^p &= \sum_{k=0}^{\infty} \left\| 1_{C_k} \int_{4^{-k}/\kappa^2}^1 |e^{-(\delta'+\delta)tL} f| \frac{dt}{t} \right\|_p^p \\ (6) \quad &\lesssim \sum_{k=0}^{\infty} \left( \int_{4^{-k}/\kappa^2}^1 \|1_{C_k} e^{-(\delta'+\delta)tL} (1_{\mathbb{R}^n \setminus C_k^*} f)\|_p \frac{dt}{t} \right)^p \\ &\quad + \sum_{k=0}^{\infty} \left\| 1_{C_k} \int_{4^{-k}/\kappa^2}^1 |e^{-(\delta'+\delta)tL} (1_{C_k^*} f)| \frac{dt}{t} \right\|_p^p. \end{aligned}$$

**Part II (off-diagonal terms):** Let  $1 \leq p \leq 2$  for the time being. We choose  $\delta, \delta' > 0$  such that  $8(\delta' + \delta) \leq 4^{-3}$  and take care of the first two annuli with  $k = 0, 1$  simply by

$$\int_{4^{-k}/\kappa^2}^1 \|1_{C_k} e^{-(\delta'+\delta)tL} (1_{\mathbb{R}^n \setminus C_k^*} f)\|_p \frac{dt}{t} \leq \left( \int_{4^{-k}/\kappa^2}^1 \frac{dt}{t} \right) \|f\|_p \lesssim (k+1) \|f\|_p.$$

For the general case with  $k \geq 2$  we write

$$\mathbb{R}^n \setminus C_k^* = B(0, 2^{k-2}) \cup \bigcup_{l=2}^{\infty} C_{k+l}.$$

Observing that  $d(C_k, B(0, 2^{k-2})) = 2^{k-2}$ , we use Lemma 3 to obtain for  $t \leq 1$  the estimate

$$\begin{aligned} \|1_{C_k} e^{-(\delta'+\delta)tL} 1_{B(0, 2^{k-2})}\|_{p \rightarrow p} &\lesssim 2^{kn} \exp\left(-\frac{4^{k-2}}{8(\delta'+\delta)t}\right) \exp\left(\frac{4^k + 4^{k-2}}{2}\right) \\ &\leq 2^{kn} \exp(-4^{k+1} + 4^k) \\ &\lesssim \exp(-4^k). \end{aligned}$$

Furthermore, since  $d(C_k, C_{k+l}) = 2^{k+l-2}$ , Lemma 3 implies that for  $t \leq 1$  we have

$$\begin{aligned} \|1_{C_k} e^{-(\delta'+\delta)tL} 1_{C_{k+l}}\|_{p \rightarrow p} &\lesssim 2^{kn} \exp\left(-\frac{4^{k+l-2}}{8(\delta'+\delta)t}\right) \exp\left(\frac{4^k + 4^{k+l}}{2}\right) \\ &\leq 2^{kn} \exp(-4^{k+l+1} + 4^{k+l}) \\ &\lesssim \exp(-4^{k+l}). \end{aligned}$$



We are now ready to estimate the off-diagonal terms for  $k \geq 2$ :

$$\begin{aligned}
 & \int_{4^{-k}/\kappa^2}^1 \|1_{C_k} e^{-(\delta'+\delta)tL} (1_{\mathbb{R}^n \setminus C_k^*} f)\|_p \frac{dt}{t} \\
 & \leq \int_{4^{-k}/\kappa^2}^1 \|1_{C_k} e^{-(\delta'+\delta)tL} (1_{B(0,2^{k-2})} f)\|_p \frac{dt}{t} \\
 & \quad + \sum_{l=2}^{\infty} \int_{4^{-k}/\kappa^2}^1 \|1_{C_k} e^{-(\delta'+\delta)tL} (1_{C_{k+l}} f)\|_p \frac{dt}{t} \\
 & \lesssim \left( \int_{4^{-k}/\kappa^2}^1 \frac{dt}{t} \right) \exp(-4^k) \|f\|_p + \left( \int_{4^{-k}/\kappa^2}^1 \frac{dt}{t} \right) \sum_{l=2}^{\infty} \exp(-4^{k+l}) \|f\|_p \\
 & \lesssim (k+1) \exp(-4^k) \|f\|_p
 \end{aligned}$$

so that the sum of the off-diagonal terms in (6) is under control

$$\sum_{k=0}^{\infty} \left( \int_{4^{-k}/\kappa^2}^1 \|1_{C_k} e^{-(\delta'+\delta)tL} (1_{\mathbb{R}^n \setminus C_k^*} f)\|_p \frac{dt}{t} \right)^p \lesssim \|f\|_p^p.$$

**Part II (on-diagonal terms):** We then consider the on-diagonal terms in (6).

Let us begin with  $p = 1$  and estimate for  $k \geq 0$  simply as follows:

$$\begin{aligned}
 \left\| 1_{C_k} \int_{4^{-k}/\kappa^2}^1 |e^{-(\delta'+\delta)tL} (1_{C_k^*} f)| \frac{dt}{t} \right\|_1 & \leq \int_{4^{-k}/\kappa^2}^1 \|1_{C_k} e^{-(\delta'+\delta)tL} (1_{C_k^*} f)\|_1 \frac{dt}{t} \\
 & \leq \left( \int_{4^{-k}/\kappa^2}^1 \frac{dt}{t} \right) \|1_{C_k^*} f\|_1 \\
 & \lesssim (k+1) \|1_{C_k^*} f\|_1.
 \end{aligned}$$

For the sum of the on-diagonal terms we then obtain

$$\sum_{k=0}^{\infty} \left\| 1_{C_k} \int_{4^{-k}/\kappa^2}^1 |e^{-(\delta'+\delta)tL} (1_{C_k^*} f)| \frac{dt}{t} \right\|_1 \lesssim \sum_{k=0}^{\infty} (k+1) \|1_{C_k^*} f\|_1 \approx \|(1 + \log_+ |\cdot|) f\|_1,$$

as required.

Let then  $p > 1$  and choose  $\kappa$  to be a power of 4 and write  $N(k) = k - 1 + 2 \log_4 \kappa$  so that  $4^{-k+N(k)+1}/\kappa^2 = 1$  for each  $k \geq 0$ . We start by partitioning the time integral as follows:

$$\begin{aligned}
 (7) \quad & \sum_{k=0}^{\infty} \left\| 1_{C_k} \int_{4^{-k}/\kappa^2}^1 |e^{-(\delta'+\delta)tL} (1_{C_k^*} f)| \frac{dt}{t} \right\|_p^p \\
 & = \sum_{k=0}^{\infty} \sum_{j=0}^{N(k)} \left\| 1_{C_k} \int_{4^{-k+j}/\kappa^2}^{4^{-k+j+1}/\kappa^2} |e^{-(\delta'+\delta)tL} (1_{C_k^*} f)| \frac{dt}{t} \right\|_p^p \\
 & \leq \sum_{k=0}^{\infty} \sum_{j=0}^{N(k)} \left( \int_{4^{-k+j}/\kappa^2}^{4^{-k+j+1}/\kappa^2} \|1_{C_k} e^{-(\delta'+\delta)tL} (1_{C_k^*} f)\|_p \frac{dt}{t} \right)^p.
 \end{aligned}$$

For each  $k, j \geq 0$  let us denote by  $q(k, j)$  the hypercontractive exponent (cf. (4)) from time  $(\delta' + \delta)4^{-k+j}/\kappa^2$ , i.e.  $q(k, j) = 1 + (p-1)e^{2(\delta'+\delta)4^{-k+j}/\kappa^2}$ . Then, using Hölder's inequality, we have for  $t \geq 4^{-k+j}/\kappa^2$ ,

$$\|1_{C_k} e^{-(\delta'+\delta)tL} (1_{C_k^*} f)\|_p \leq \gamma(C_k)^{\frac{1}{p} - \frac{1}{q(k,j)}} \|1_{C_k} e^{-(\delta'+\delta)tL} (1_{C_k^*} f)\|_{q(k,j)} \lesssim e^{-c4^j} \|1_{C_k^*} f\|_p,$$

where the decay factor from the last inequality will be justified next. Firstly,

$$\gamma(C_k) \lesssim \int_{2^{k-1}}^{\infty} e^{r^2} r^{n-1} dr \lesssim e^{-c4^k}.$$

Secondly,

$$\frac{1}{p} - \frac{1}{q(k, j)} = \frac{p-1}{p} \frac{e^{2(\delta'+\delta)4^{-k+j}/\kappa^2} - 1}{1 + (p-1)e^{2(\delta'+\delta)4^{-k+j}/\kappa^2}} \gtrsim e^{2(\delta'+\delta)4^{-k+j}/\kappa^2} - 1 \gtrsim 4^{-k+j}.$$

Hence,

$$\gamma(C_k)^{\frac{1}{p} - \frac{1}{q(k, j)}} \lesssim (e^{-c4^k})^{4^{-k+j}} \lesssim e^{-c4^j},$$

as was claimed.

Returning to the sum of the on-diagonal terms in (7),

$$\begin{aligned} & \sum_{k=0}^{\infty} \sum_{j=0}^{N(k)} \left( \int_{4^{-k+j}/\kappa^2}^{4^{-k+j+1}/\kappa^2} \|1_{C_k} e^{-(\delta'+\delta)tL} (1_{C_k^*} f)\|_p \frac{dt}{t} \right)^p \\ & \lesssim \sum_{k=0}^{\infty} \|1_{C_k^*} f\|_p^p \sum_{j=0}^{N(k)} \left( \int_{4^{-k+j}/\kappa^2}^{4^{-k+j+1}/\kappa^2} \frac{dt}{t} \right)^p e^{-cp4^j} \\ & \lesssim \sum_{k=0}^{\infty} \|1_{C_k^*} f\|_p^p \lesssim \|f\|_p^p. \end{aligned}$$

This finishes the proof.  $\square$

REMARK. As is clear from the proof above, if one could show that there exists an  $\alpha > 1$  such that for all  $k \geq 0$  and all  $0 \leq j \leq N(k)$ ,

$$\|1_{C_k} e^{-tL} (1_{C_k^*} f)\|_1 \lesssim j^{-\alpha} \|1_{C_k^*} f\|_1, \quad t \gtrsim 4^{-k+j},$$

then the desired inequality  $\|\pi_3 f\|_1 \lesssim \|f\|_1$  would follow (for multipliers satisfying Condition D).

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